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HARMONIC NON-LINEAR RESPONSE OF BECK'S COLUMN TO A LATERAL EXCITATION

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Abstract—The non-linear response of a column with a follower force (Beck's column) subjected to a distributed periodic lateral excitation, or to a support excitation, is determined. An analytical solution for the response amplitude in terms of the loading and system parameters is obtained by a perturbation analysis of the differential equations of motion. Non-linear inertia and non-linear curvature terms are taken into account in the formulation of the differential equations.

INTRODUCTION

A cantilever column subject to a compressive tangential force P applied at its free end is known in the literature as Beck's column, after Beck[1] in 1952 correctly calculated the critical load P_{cr} for such column. Since the virtual work associated with the follower force P cannot be obtained from a potential function, the problem solved by Beck is non-conservative.

The static stability criterion does not yield the correct value for the critical load P_{cr} for Beck's column. Beck was able to obtain the correct value for P_{cr} by investigating the free vibrations of the column under the action of the load P.

The static and dynamic methods for treating elastic stability problems, and Beck's problem, are discussed in a recent book by Simitses[2]. A classical book by Bolotin[3] is devoted entirely to the study of non-conservative problems of the theory of elastic stability, and a monograph by Herrmann[4] presents a variety of examples and an extensive bibliography in the area of dynamics and stability of mechanical systems with follower forces. The applicability and limitations of the static method for analyzing conservative and non-conservative problems in structural stability, also illustrated by a variety of examples, is discussed in detail by Leipholz[5]. Several models demonstrating instability of equilibrium in systems with follower forces have been offered by Herrmann[4], and by Herrmann, Nemat-Nasser and Prasad[6].

The analysis of the linear dynamical behavior, directed mainly to the determination of the critical load for columns subjected to non-conservative forces, and the analysis of discrete models for such columns, has received much attention in the literature. Prior to the publication of his monograph previously mentioned, an earlier review of the subject has been presented by Herrmann[7]. The stability of a two degree-of-freedom model of Beck's column has been studied by Ziegler[8]. Herrmann and Bungay[9], by allowing the direction of the tangential load in Ziegler's model to change, further clarified the relationship between applicability of static and dynamic stability criteria and the non-conservative character of the load.

The dynamics of the model recently analyzed by Herrmann and Bungay in[9], with emphasis on the determination of its critical load, has been recently re-considered by Ku[10] with the introduction of an internal constraint into the system; Sundararajan[11, 12] determined the influence on the natural frequency and on the critical load for Beck's column, and for its discrete version, caused by a linear spring support at the point of application of the tangential load.

The quantitative effect of viscous damping on the critical load for non-conservative systems was investigated by Nemat-Nasser, Prasad and Herrmann[13] and by Plaut and Infante[14]. It was shown in[14] that the critical load for Beck's column increases with increased damping to a limiting value that is nearly twice the critical load for the undamped case.

Recent studies concerning the dynamical behavior of elastic non-conservative systems with

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two independent loading parameters have been recently considered by McGill[15] who used the two-term approximation of Galerkin's method to investigate the instability of a cantilever column subject to distributed vertical and tangential forces; by Huseyin and Plaut[16] for a simply supported column subjected to a concentrated constant force and distributed tangential forces; by Sugiyama and Kawagoe[17] for columns with six types of boundary conditions, and subjected to uniformly distributed vertical and tangential forces; by Celep[18] for a narrow rectangular strip with asymmetric support and subjected to uniformly distributed vertical and follower forces.

Although the non-linear dynamics of beams has received much attention in the literature as exemplified by the works of Haight and King[19], Ho, Scott and Eisley[20, 21], Nayfeh[22], Atluri[23], Nayfeh, Mook and Sridhar[24], Woodall[25], Rehfield[26] and Crespo da Silva and Glynn[27–28], this is not the case for beam-columns, in particular when a non-conservative load is present such as in Beck's column. For the latter case, attention has been mostly focused on the linear dynamical behavior and the determination of the critical load for the system. The author's purpose is to address the moderately large non-linear motion of Beck's column by taking into account the geometric non-linearities present in the system.

In the present paper the non-linear oscillations of Beck's column subjected to a "small" distributed periodic lateral excitation that maintains a fixed direction in space are analyzed. Damping of the lateral motion is assumed to be viscous, as considered in[14], but "small" in a sense that is clarified later. Non-linear equations of motion are derived via Hamilton's principle; the equations obtained contain not only the familiar non-linear inertia terms but also terms due to non-linear curvature.

NON-LINEAR EQUATIONS OF MOTION AND SIMPLIFYING ASSUMPTIONS

Consider a uniform and compact clamped-free column of constant length L, and mass m per unit length, subject to a constant tangential compressive force P applied at the free end, and to a periodic planar lateral distributed load $Q(s, t) = f(s)\cos\Omega t$; here, s denotes the arc-length along the column measured from the clamped end ($s = 0$), and t is time (Fig. 1). In its undeformed state the column is assumed to be colinear with the inertial x-axis shown in Fig. 1. The elastic displacements of the centroid of the cross section S (normal to the neutral axis) at an arbitrary station s of the column are denoted by $u(s, t)$ and $v(s, t)$; they are also shown in Fig. 1. In the deformed state, the angle between the inertial x-axis and the tangent to the column's neutral axis at s is denoted by ψ (see Fig. 1); this angle is related to the spatial derivatives of the deformations u and v as

$$\tan\psi = (\partial v/\partial s)/[1 + \partial u/\partial s]. \tag{1}$$

By assuming the column to be inextensional, it follows that

$$(1 + \partial u/\partial s)^2 + (\partial v/\partial s)^2 - 1 = 0. \tag{2}$$

In the development to follow it will also be assumed that: (1) the strains are small enough for the material to behave linearly in the sense that the bending moment at station s is proportional to the bending curvature $\partial\psi/\partial s$ at s (which is non-linear in $\partial u/\partial s$ and $\partial v/\partial s$); (2) plane cross sections before bending remain plane after bending and (3) damping of the flexural motion is viscous, with damping coefficient c and virtual work equal to $-c(\partial v/\partial t)\delta v$. By neglecting the effects of the distributed mass moments of inertia on the motion of the column, as well as the shear energy when compared to the bending energy, Hamilton's principle, with the constraint equation (2) adjoined to the Lagrangian of the motion by means of a Lagrange multiplier λ, requires that[29]

$$\delta I = \delta \int_{s=0}^{L}\int_{t_1}^{t_2}\left\{\frac{1}{2}m(\dot{u}^2+\dot{v}^2) - \frac{1}{2}D_\zeta\psi'^2 + \frac{\lambda}{2}[1-(1+u')^2 - v'^2]\right\}\mathrm{d}s\,\mathrm{d}t$$
$$-\int_{s=0}^{L}\int_{t_1}^{t_2}\{c\dot{v}\delta v + Q(s,t)\delta v\}\,\mathrm{d}s\,\mathrm{d}t + \int_{t_1}^{t_2}\delta W_B\,\mathrm{d}t = 0. \tag{3}$$

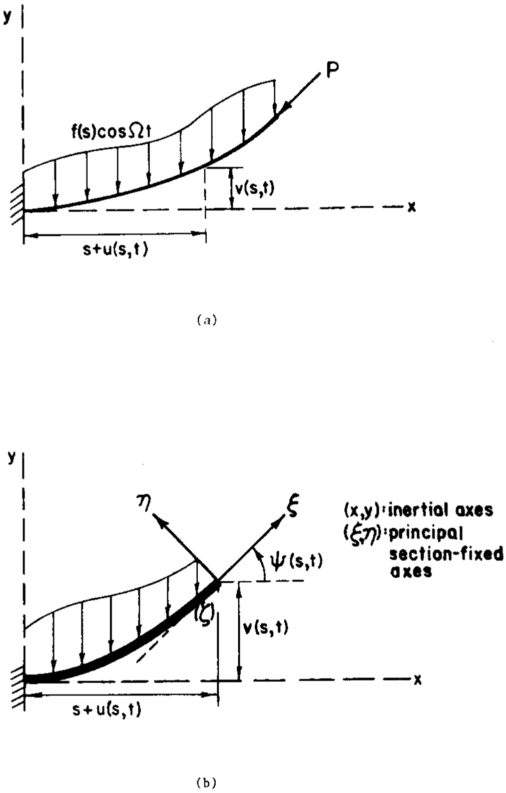

Fig. 1. Beck's column subjected to a planar laterally-distributed periodic excitation.

In eqn (3) dots and primes denote, respectively, partial differentiations with respect to time t and arc-length s, and D_ζ is the bending stiffness of the column (the ζ-axis is normal to the ξ and η axes shown in Fig. 1). The term δW_B is the virtual work associated with the follower force applied at the end $s = L$. By letting $\hat{\mathbf{x}}$ and $\hat{\mathbf{y}}$ denote the unit vectors in the inertial x and y directions shown in Fig. 1, and by noticing that eqns (1) and (2) imply that $\cos\psi = 1 + u'$ and $\sin\psi = v'$, the virtual work δW_B associated with the follower force P, is obtained as

$$\begin{aligned}\delta W_B &= \{-P[(\cos\psi)\hat{\mathbf{x}} + (\sin\psi)\hat{\mathbf{y}}]\cdot[(\delta u)\hat{\mathbf{x}} + (\delta v)\hat{\mathbf{y}}]\}_{s=L} \\ &= -P[(1+u')\delta u + v'\delta v]_{s=L}. \end{aligned} \tag{4}$$

By integrating by parts the terms in δI that contain the variations $\delta\dot{u}$, $\delta\dot{v}$, $\delta\psi'$, $\delta u'$ and $\delta v'$ that result when the variation of the first double integral in eqn (3) is taken, one obtains by using eqn (4)

$$\delta I = \int_{s=0}^{L}\int_{t_1}^{t_2}\{[(\lambda(1+u'))' - m\ddot{u}]\delta u + [(\lambda v')' - m\ddot{v} - c\dot{v} - Q(s,t)]\delta v$$

$$+ (D_\zeta\psi')'\delta\psi\}\,\mathrm{d}s\,\mathrm{d}t - \int_{t_1}^{t_2}\{D_\zeta\psi'\,\delta\psi + (\lambda+P)(1+u')\delta u \; + P)v'\delta v\}_{s=0}^{L}\,\mathrm{d}t = 0. \tag{5}$$

By making use of eqns (1) and (2), the relation between the variations $\delta\psi$, δu and δv is obtained as

$$\delta\psi = (\partial\psi/\partial u')\delta u' + (\partial\psi/\partial v')\delta v' = -v'\delta u' + (1+u')\delta v'. \tag{6}$$

By combining eqns (5) and (6) the following differential equations of motion, and boundary condition, are then obtained,

$$\{v'(D_\zeta\psi')' + \lambda(1+u')\}' = G_u' = m\ddot{u} \tag{7a}$$

$$\{-(1+u')(D_\zeta\psi')' + \lambda v'\}' = G'_v = m\ddot{v} + c\dot{v} + f(s)\cos\Omega t \tag{7b}$$

$$(1+u')^2 + v'^2 - 1 = 0 \tag{7c}$$

$$\{-[G_u + P(1+u')]\delta u - [G_v + Pv']\delta v + D_\zeta\psi'[v'\delta u' - (1+u')\delta v']\}_{s=0}^{L} = 0. \tag{8}$$

To analyze the motion described by eqns (7a–c) and (8) analytically, by perturbation techniques, the non-linearities in these equations will now be expanded in a Taylor series about the straight equilibrium state $v = \psi = 0$. By retaining in the expansions non-linear terms up to third-order, eqns (7c) and (1) yield for u' and ψ, respectively,

$$u' \approx -v'^2/2 \tag{9a}$$

$$\psi \approx v' + v'^3/6. \tag{9b}$$

Direct substitution of eqns (9a–b) into eqns (7a–b) and (8) yield the following differential equations and boundary condition with order-three polynomial non-linearities, under the assumption that the stiffness D_ζ is constant,

$$\{D_\zeta v'v''' + \lambda(1 - v'^2/2)\}' = G'_u = m\ddot{u} \tag{10a}$$

$$\{-D_\zeta(v''' + v'v''^2) + \lambda v'\}' = G'_v = m\ddot{v} + c\dot{v} + f(s)\cos\Omega t \tag{10b}$$

$$\{[G_u + P(1 - v'^2/2)]\delta u + [G_v + Pv']\delta v - D_\zeta v''(1 + v'^2)\delta v'\}_{s=0}^{L} = 0. \tag{11}$$

The longitudinal elastic displacement $u(s,t)$ in eqn (10a) can be obtained by integration of eqn (9a).

The physical boundary conditions for the clamped-free column shown in Fig. 1, and the natural boundary conditions obtained from eqn (11) are

$$u(0,t) = v(0,t) = v'(0,t) = v''(L,t) = 0 \tag{12a}$$

$$\{G_u + P(1 - v'^2/2)\}_{s=L} = 0 \tag{12b}$$

$$\{G_v + Pv'\}_{s=L} = 0. \tag{12c}$$

The boundary conditions given by eqns (12a–c), and eqn (9a), can be used to reduce eqns (10a–b) to single integro-differential equation in the flexural displacement $v(s,t)$; the latter equation will prove to be more convenient to use to analyze the flexural motion.

To eliminate the elastic displacement $u(s,t)$ and the Lagrange multiplier λ from eqns (10a–b) use is made of the boundary conditions $u(0,t) = [G_u + P(1 - v'^2/2)]_{s=L} = 0$ and eqns (9a) and (10a) to obtain, to order-three,

$$u(s,t) = -\frac{1}{2}\int_0^s v'^2\,ds \tag{13a}$$

$$\lambda(s,t) = -P\left\{1 + \frac{1}{2}[v'^2(s,t) - v'^2(L,t)]\right\} - D_\zeta v'''v' - \frac{m}{2}\int_L^s\int_0^s (v'^2\,ds)^{\cdot\cdot}\,ds. \tag{13b}$$

With $\lambda(s,t)$ given by eqn (13b), the differential equation (10b) for the flexural displacement $v(s,t)$ can be written in integro-differential form as,

$$m\ddot{v} + c\dot{v} + Pv'' + D_\zeta v'''' = -\{D_\zeta v'(v'v'')'\}' - f(s)\cos\Omega t$$
$$-\frac{m}{2}\left\{v'\int_L^s\int_0^s (v'^2\,ds)^{\cdot\cdot}ds\right\}' + \frac{P}{2}\{v'[v'^2(L,t) - v'^2(s,t)]\}'. \tag{14}$$

Before proceeding to the next section, where the non-linear motion $v(s,t)$ is analyzed, eqn

(14) is written in non-dimensional form by defining

$$s^* = s/L; \qquad v^* = v/L \tag{15a}$$

$$P^* = PL^2/D_\zeta; \qquad c^* = c[D_\zeta/(mL^2)]^{1/2} \tag{15b}$$

$$t^* = t[D_\zeta/(mL^4)]^{1/2} \tag{15c}$$

$$f^*(s) = f(s)L^3/D_\zeta; \qquad \Omega^* = \Omega[(mL^4)/D_\zeta]^{1/2}. \tag{15d}$$

In terms of the non-dimensional variables defined by eqns (15a–d), the integro-differential equation (14) and the boundary conditions given by eqns (12a–c) are finally written as shown in eqns (16a–c) that follow; in these last equations, as well as in the subsequent sections, the star superscript has been dropped, for simplicity. It should be understood, however, that all variables written from now on are non-dimensional variables.

$$\ddot{v} + c\dot{v} + Pv'' + v'''' = -\{v'(v'v'')'\}' - f(s)\cos\Omega t$$
$$-\frac{1}{2}\left\{v'\int_1^s\int_0^s (v'^2\,ds)^{\cdot\cdot}\,ds\right\}' + \frac{P}{2}\left\{v'[v'^2(1,t) - v'^2(s,t)]\right\}' \tag{16a}$$

$$v(0,t) = v'(0,t) = v''(1,t) = 0 \tag{16b}$$

$$v'''(1,t) = 0. \tag{16c}$$

The boundary condition $v'''(1,t) = 0$ is obtained directly from eqn (12c) with G_v and λ given in eqns (10b) and (13b). It is of interest to note that eqn (16a) also applies to a Beck column supported by an oscillating base. In such case, $f(s)$ is a constant equal to the amplitude of the base oscillation and $v(s,t)$ is the transverse elastic displacement measured from the instantaneous position of the column's neutral axis.

ANALYSIS OF THE RESONANT FLEXURAL MOTION

To analyze the motion described by eqns (16a–c) the method of multiple time scales [30] will be used. For this, an arbitrary small parameter ϵ, and two time scales $t_0 = t$ and $t_2 = \epsilon^2 t$ are introduced. The response v, and the boundary conditions, are expanded as

$$v(s,t;\epsilon) = \epsilon v_1(s,t_0,t_2) + \epsilon^3 v_3(s,t_0,t_2) + \ldots. \tag{17}$$

Also, the damping coefficient c and the spatial variation $f(s)$ of the forcing function are assumed to be small and are expressed as

$$c = \epsilon^2 c_2; \qquad f(s) = \epsilon^3 \bar{f}_3(s). \tag{18}$$

By noticing that the time derivatives $\partial^n(\ )/\partial t^n$ $(n = 1, 2)$ in eqn (16a) are now transformed as $\partial v/\partial t = d_0 v + \epsilon^2 d_2 v + \ldots$, $\partial^2 v/\partial t^2 = d_0^2 v + 2\epsilon^2 d_0 d_2 v + \ldots$, where $d_n = \partial/\partial t_n$, substitution of eqns (17) and (18) into eqns (16a–c) yield by equating to zero the coefficients of the $0(\epsilon)$ and $0(\epsilon^3)$ terms:

Order ϵ

$$d_0^2 v_1 + Pv_1'' + v_1'''' = 0 \tag{19a}$$

$$v_1(0,t_0,t_2) = v_1'(0,t_0,t_2) = v_1''(1,t_0,t_2) = v_1'''(1,t_0,t_2) = 0. \tag{19b}$$

Order ϵ^3

$$d_0^2 v_3 + Pv_3'' + v_3'''' = -2d_0 d_2 v_1 - \{(v_1'(v_1'v_1'')'\}' - c_2 d_0 v_1 - \frac{1}{2}\left\{v_1'\int_1^s\int_0^s d_0^2(v_1'^2)\,ds\,ds\right\}'$$
$$+\frac{P}{2}\{v_1'[v_1'^2(1,t_0,t_2) - v_1'^2(s,t_0,t_2)]\}' - \bar{f}_3(s)\cos\Omega t_0 \tag{20a}$$

$$v_3(0,t_0,t_2) = v_3'(0,t_0,t_2) = v_3''(1,t_0,t_2) = v_3'''(1,t_0,t_2) = 0. \tag{20b}$$

The determination of the critical load, P_{cr}, for the problem described by the differential equation (19a) with the boundary conditions given by eqn (19b) constitutes the problem addressed and solved by Beck[1].

By considering one mode of oscillation in the solution to eqn (19a), the solution $v_1(s, t_0, t_2)$ is obtained as,

$$v_1(s, t_0, t_2) = F(s)A(t_2)\cos[\omega t_0 + B(t_2)] = F(s)A(t_2)\cos\psi \tag{21a}$$

where $F(s)$ is obtained by solving the differential equation

$$F''''(s) + PF''(s) - \omega^2 F(s) = 0 \tag{21b}$$

with the boundary conditions

$$F(0) = F'(0) = F''(1) = F'''(1) = 0. \tag{21c}$$

The solution to eqn (21b) satisfying the above boundary conditions is obtained as†

$$F(s) = C\left\{\sin(r_1 s) - \frac{r_1}{r_2}\sinh(r_2 s) - K[\cos(r_1 s) - \cosh(r_2 s)]\right\} \tag{22a}$$

where

$$K = (r_1^2 \sin r_1 + r_1 r_2 \sinh r_2)/(r_1^2 \cos r_1 + r_2^2 \cosh r_2). \tag{22b}$$

The constants r_1, r_2, the frequency $\omega = r_1 r_2$ and the magnitude P of the follower force are related as

$$r_1^2 = [(P^2 + 4\omega^2)^{1/2} + P]/2; \qquad r_2^2 = [(P^2 + 4\omega^2)^{1/2} - P]/2. \tag{22c}$$

With r_1 and r_2 given by eqn (22c), the frequency ω is obtained by solving the characteristic equation

$$P^2 = \omega P(\sin r_1)\sinh r_2 + 2\omega^2[1 + (\cos r_1)\cosh r_2] = 0. \tag{22d}$$

Figure 2, obtained by solving eqn (22d) numerically, shows the variation of the natural frequency of oscillation, ω, as a function of P. The critical load, $P_{cr} \approx 20.05$ as obtained by Beck, corresponds to point B in that figure.

To determine the response of the system to the lateral excitation $\bar{f}_3(s)\cos\Omega t_0$, the solution $v_1(s, t_0, t_2)$ given by eqn (21a) is substituted into eqn (20a) to obtain, with dots now denoting partial differentiation with respect to the slow time scale t_2,

$$\begin{aligned} d_0^2 v_3 + Pv_3'' + v_3'''' &= (2\omega\dot{A} + c_2\omega A)F(s)\sin\psi + 2\omega A\dot{B}F(s)\cos\psi \\ &+ \frac{A^3}{4}\left\{\frac{P}{2}F'[F'^2(1) - F'^2(s)] - F'(F'F'')'\right\}'(3\cos\psi + \cos 3\psi) \\ &+ \frac{\omega^2}{2}A^3\left\{F'\int_1^s\int_0^s F'^2\,ds\,ds\right\}'(\cos\psi + \cos 3\psi) - \bar{f}_3(s)\cos\Omega t_0. \end{aligned} \tag{23}$$

An approximate solution for the amplitude $A(t_2)$ and phase $B(t_2)$ of the response $\epsilon v_1(s, t_0, t_2)$ can be obtained by applying Galerkin's method[5, 29] to eqn (23). Toward this end, by considering only one mode in the response, we let

$$v_3(s, t_0, t_2) \approx F(s)v_{3t}(t_0, t_2) \tag{24}$$

which satisfies all the boundary conditions given by eqn (20b).

†The constant $C \neq 0$ in eqn (22a) is arbitrary; its value will be chosen later so that $\int_0^1 F^2(s)\,ds = 1$, for convenience.

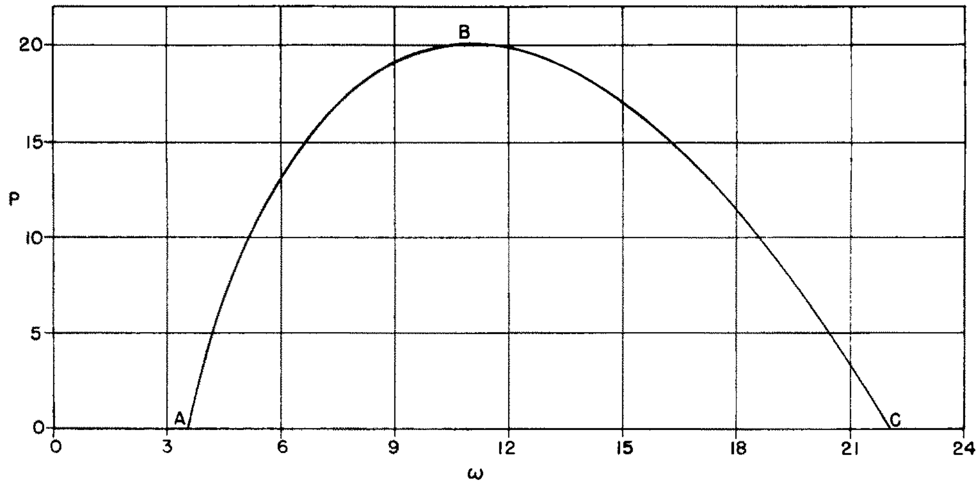

Fig. 2. Natural frequency ω vs the magnitude P of the follower force.

By multiplying both sides of eqn (23) by the eigenfunction $F(s)$ given by eqn (22a), and subsequently integrating over the domain of s, an ordinary differential equation is obtained for $v_{3t}(t_0, t_2)$. First, we choose the constant C in eqn (22a) so that $\int_0^1 F^2(s)\,ds = 1$, and define the following parameters,

$$\alpha_3 = \int_0^1 F(s)\left\{F'(s)[F'(s)F''(s)]' + \frac{P}{2}F'(s)[F'^2(s) - F'^2(1)]\right\}'\,ds \tag{25a}$$

$$\alpha_4 = \int_0^1 F(s)\left\{F'(s)\int_1^s \int_0^s F'^2(s)\,ds\,ds\right\}'\,ds \tag{25b}$$

$$f_3 = \int_0^1 F(s)\,\tilde{f}_3(s)\,ds. \tag{25c}$$

By making use of eqns (21b–c), eqn (23) yields

$$d_0^2 v_{3t} + Pv_{3t}\int_0^1 F(s)F''(s)\,ds + v_{3t}\int_0^1 F(s)F''''(s)\,ds = d_0^2 v_{3t} + \omega^2 v_{3t} = \omega(2\dot{A} + c_2 A)\sin\psi$$
$$+\left[2\omega A\dot{B} - \frac{3}{4}\alpha_3 A^3 + \frac{1}{2}\omega^2\alpha_4 A^3\right]\cos\psi - f_3\cos\Omega t_0 + \mathrm{NST} \tag{26}$$

where NST stands for "non-secular terms", namely, the terms of frequency 3ω.

When the driving frequency Ω is not "near" the natural frequency of the column, ω, the behavior of the motion can be approximately described by the solution of the $0(\epsilon)$ equations by making $f(s) = 0(\epsilon)$. However, when Ω is "near" the natural frequency ω, the non-linearities in the equation of motion will play a significant role in determining the response of the system. To address this latter case, we express the nearness of Ω to ω by introducing a normalized detuning $\epsilon^2\sigma_2$ by writing,

$$\Omega = \omega(1 + \epsilon^2\sigma_2). \tag{27}$$

By noticing that eqn (27) implies that the forcing function $f_3\cos\Omega t_0$ in the r.h.s. of eqn (26) can be written as

$$f_3\cos\Omega t_0 = f_3\cos(\omega t_0 + \omega\sigma_2 t_2) = f_3\cos(\psi + \omega\sigma_2 t_2 - B) = f_3\cos(\psi + \phi) \tag{28}$$

elimination of the secular terms in eqn (26), which is the condition for obtaining a periodic solution for v_{3t}, yield the following differential equations for the amplitude $A(t_2)$ and phase

$B(t_2)$ of the first approximation ϵv_1

$$2\omega\dot{A} + \omega c_2 A + f_3 \sin\phi = 0 \tag{29a}$$

$$A[2\omega(\omega\sigma_2 - \dot{\phi}) + A^2(2\omega^2\alpha_4 - 3\alpha_3)/4] - f_3\cos\phi = 0. \tag{29b}$$

Periodic oscillations of the column correspond to the equilibrium solutions $\dot{A}_e = \dot{\phi}_e = 0$ of eqns (29a–b). By eliminating the angle ϕ_e from eqns (29) the following amplitude-detuning relationship can be obtained,

$$\epsilon^2\sigma_2 = \{\pm[(\epsilon^3 f_3)^2/(\epsilon A_e)^2 - (\omega\epsilon^2 c_2)^2]^{(1/2)} - \alpha_5(1-\alpha_6)(\epsilon A_e)^2\}/(2\omega^2) \tag{30a}$$

where α_5 and α_6 are constants defined as,

$$\alpha_5 = \omega^2\alpha_4/2 - \alpha_3/4; \qquad \alpha_6 = \alpha_3/(2\alpha_5). \tag{30b}$$

It is of interest to note that the amplitude-frequency response given by eqn (30a) is of the same form as that determined by Crespo da Silva and Glynn[28] for beams with other boundary conditions and no follower force (i.e. $P = 0$). The influence of the follower force P is reflected in the constant α_3 defined by eqn (25a) and, of course, in the eigenfunction $F(s)$.

It also should be noted that if $\alpha_6 = 1$, eqn (30a) is the same as that obtained for the amplitude-frequency relationship for the motion corresponding to the linearized form of eqn (16a) with $v(s, t) \approx F(s)A_{\text{lin}} \cdot \cos(\omega t + \alpha)$ and $\Omega \approx \omega$; the amplitude of the linear response of the column is obtained, by using Galerkin's method, as

$$A_{\text{lin}}^2[(\Omega c)^2 + (\omega^2 - \Omega^2)^2] = \left[\int_0^1 F(s)f(s)\,\mathrm{d}s\right]^2. \tag{31}$$

Figures 3 and 4 show the variation of the constants α_5 and α_6 as a function of the magnitude P of the follower force for the straight column. The portions AB and BC of these figures correspond, respectively, to the portions AB and BC of the (ω, P) curve shown in Fig. 2. The values of $\alpha_5(P)$ and $\alpha_6(P)$ in Figs. 3 and 4 were obtained by numerical evaluation of the integrals given in eqns (25a–b).

The stability of the harmonic oscillation of amplitude $\epsilon A_e(P, \Omega; \epsilon^3 f_3, \epsilon^2 c_2)$ given by eqn (30a) can be ascertained by letting

$$A = A_e + A_s; \qquad \phi = \phi_e + \phi_s \tag{32}$$

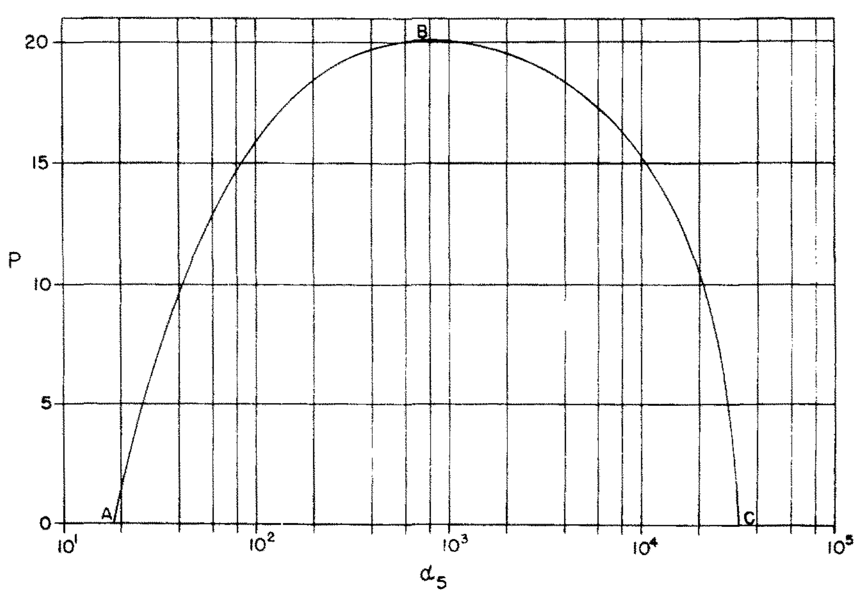

Fig. 3. Dependence of α_5 on the magnitude P of the follower force.

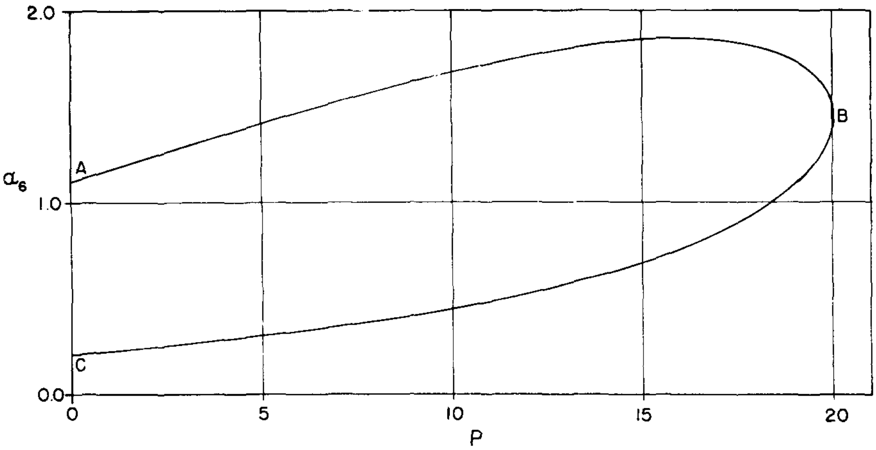

Fig. 4. Dependence of α_6 on the magnitude P of the follower force.

and linearizing the differential equations (29a, b) in A_s and ϕ_s. By letting $A_s \propto \exp(\mu t_2)$ and $\phi_s \propto \exp(\mu t_2)$ in these linear differential equations, it is found that μ must satisfy the following characteristic equation

$$\mu^2 + c_2\mu + g/(4\omega^2) = 0 \tag{33a}$$

with

$$g = (\omega c)^2 + [2\omega^2\sigma_2 + \alpha_5(1-\alpha_6)A_e^2][2\omega^2\sigma_2 + 3\alpha_5(1-\alpha_6)A_e^2]. \tag{33b}$$

The equilibrium described by eqn (30a) is stable when $g > 0$ (and, of course, $c_2 > 0$).

Figures 5 and 6 show the amplitude-frequency response curve for the column for several values of the magnitude P of the follower force and for $\epsilon^2 c_2 = 0.05$ and $\epsilon^3 f_3 = 0.03$. Figure 5 shows the response "amplitude" ϵA_e as P is increased along the first branch AB of the ω-vs-P curve shown in Fig. 2 while Fig. 6 shows the response "amplitude" as P takes values along the

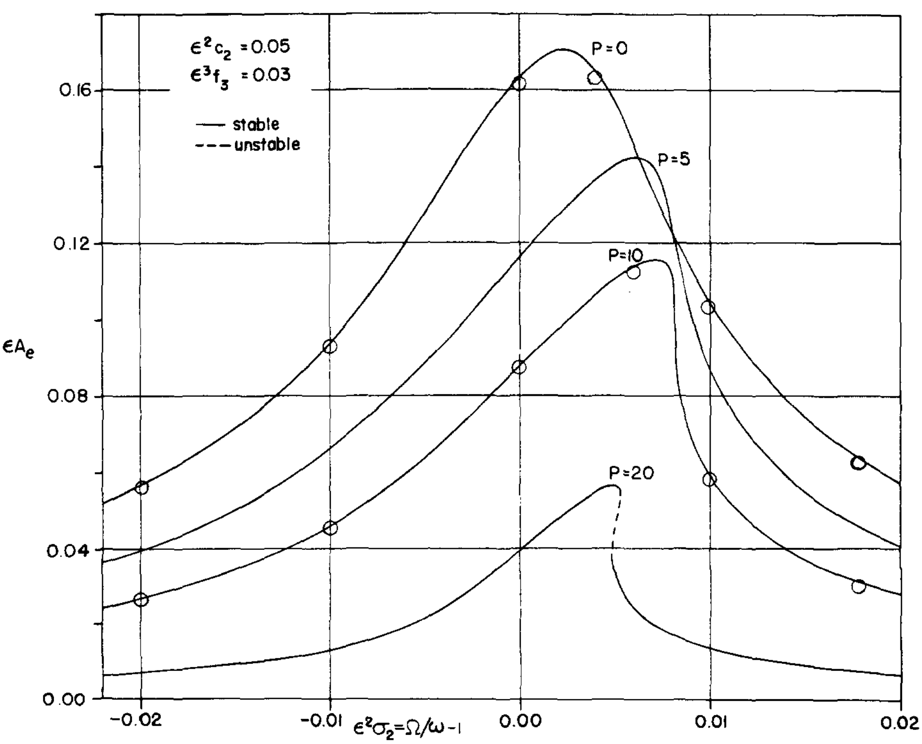

Fig. 5. Amplitude-frequency non-linear response of Beck's column to a periodic lateral excitation.

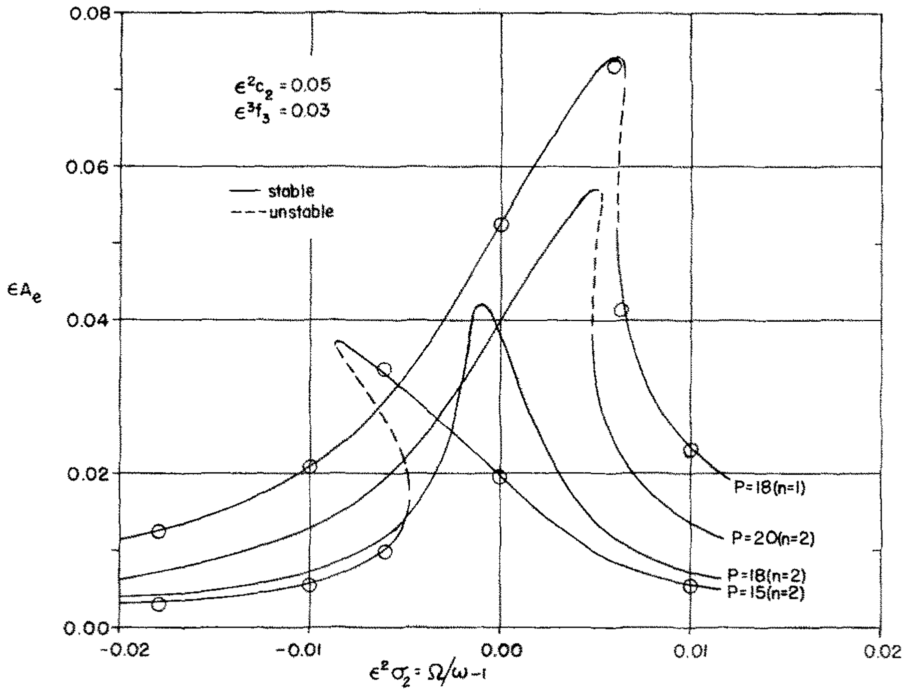

Fig. 6. Amplitude-frequency non-linear response of Beck's column to a periodic lateral excitation ($n = 1$: branch AB in Fig. 2; $n = 2$: branch BC in Fig. 2).

branches AB ($n = 1$ curve) and BC ($n = 2$ curves) of Fig. 2. The stable and unstable portions of the response are shown in those figures. Points marked with a circle in those figures were obtained by numerical integration of eqn (16a) after it has been transformed to an ordinary differential equation by using Galerkin's method with $v(s, t) \approx F(s)V(t)$.

CONCLUDING REMARKS

As evidenced by Figs. 5 and 6 as P is changed along the branch ABC of the ω-vs-P curve shown in Fig. 2, with the driving frequency taking values Ω that are close to the natural frequency ω, a jump phenomenon occur for certain ranges of values of P; as the pair of values (P, Ω) is changed in the direction of increasing natural frequency ω in Fig. 2, the non-linearities in the system change from hardening to softening. This change is accompanied by a decrease in the value of the parameter α_6 defined by eqn (30b). As the value of α_6 is decreased the influence

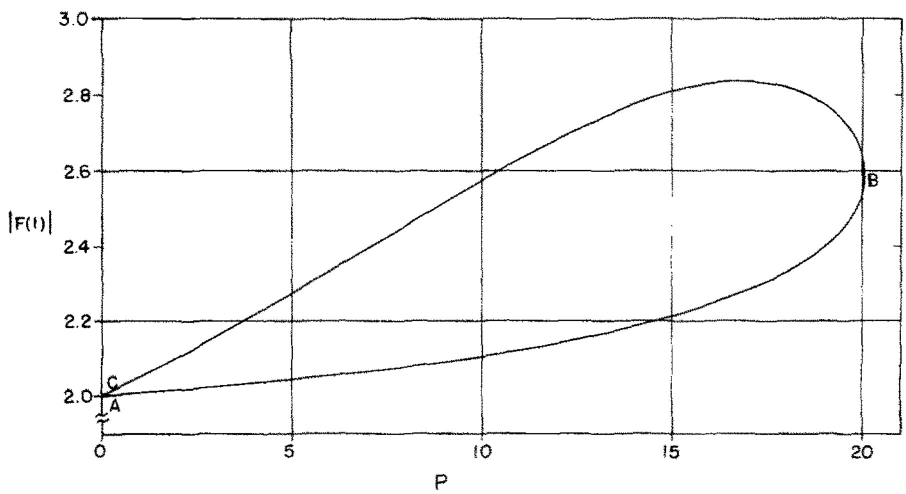

Fig. 7. Absolute value of the eigenfunction $F(s)$ at $s = 1$, as a function of the magnitude P of the follower force.

of the non-linear inertia term [i.e. the double integral term in eqn (16a)] becomes more dominant than the curvature non-linearities.

As disclosed from eqns (30a) and (21a) the maximum value of the amplitude of oscillation, at the tip of the column, is given as $(\epsilon v_1)_{max} = F(s = 1)\ \epsilon^3 f_3/(\omega\epsilon^2 c_2)$. The variation of the absolute value of $F(s = 1)$ as a function of the magnitude P of the follower force is shown in Fig. 7. It should be noted that if the motion of the column is not constrained to be planar, stable points in Figs. 5 and 6 may turn out to be unstable due to non-linear coupling between $v(s, t)$ and out-of-plane elastic displacements. An investigation of this phenomenon is presented by the author elsewhere[31].

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